PROPAGATION OF DEM UNCERTAINTY: AN INTERVAL ARITHMETIC APPROACH.

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ABSTRACT

In several GIS applications it is necessary to classify the topographic surface represented by a Digital Elevation Model (DEM) into specific slope classes. However, the computation of these classes from a DEM is influenced by the uncertainty in the elevations of those models. Monte-Carlo method has been used to study the effect of DEM uncertainty on topographic parameters. This method requires intensive computation, which turns the implementation very hard in most GIS software packages. Interval Arithmetic (IA) has been used as a successfully alternative to the Monte-Carlo method. Intervals of variation are used instead of a significant number of simulations. In this paper, we will show that the use of the IA constitutes a valid alternative to study the propagation of the uncertainty. Comparing the results from both methods, we conclude that the propagation of DEM uncertainty to the calculation of slope classes using IA is preferable to Monte-Carlo method, since it does not require so intensive computations, which turns the implementation easier in GIS analyses.

INTRODUCTION

The classification of the topographical surface in slope classes assumes an important role in several spatial analyses performed by GIS tools. The results of these analyses are frequently critical in decision making, which can affect many sectors in economic and social activities. Delimitation of ecological protection zones, identification of areas for urban development and the cartography of geohazard zones are some examples where slope is an important parameter.

However, slope classification is affected by elevation uncertainty in DEM which is generated by errors in the acquisition of topographical data and in the interpolation methods used to build the elevation model. In this paper, we will go to use the word uncertainty to express the lack of knowledge about the true value of some quantity. The uncertainty includes errors or uncertainties due to imperfections of measurement systems and also the effect of the cartographic generalization which can not be avoid in cartographic modeling. In order that information quality about topography can be accessed by GIS users during spatial analysis, the following procedure can be executed:

- build an elevation uncertainty model;
- propagate that uncertainty to derived terrain features (slope, aspect, curvature…);
- specify appropriate methods for uncertainty evaluation, including visualization.

The Monte-Carlo method is usually applied to study the uncertainty propagation from DEMs to derived slope classes ([10], [11], [11] e [13]). However, this method needs a large amount of computation efforts, which turns it very hard to be used in GIS applications. Interval arithmetic has been successfully used as and alternative to Monte-Carlo method ([17]). Instead to generate a certain number of simulations for some probability distribution which parameters has been estimated, interval arithmetic uses intervals of possible variation. Operations from interval arithmetic can be used to process that type of data ([15] e [16]). Therefore, interval arithmetic is an appropriated tool to deal with the uncertainty of spatial analysis with DEM's, giving intervals which express the propagated uncertainty.

This work will show that interval arithmetic is a good alternative to study uncertainty propagation on this type of problems. First a common uncertainty model is used for the cartographic terrain representation. That model states that for a terrain representation based on contour lines and point elevations, 90% of control point elevations should have a mean squared error not bigger than half of the contour interval. It is used a specific interpolator (i.e., elastic grid), to generate a DEM from contours, to ensure the equivalence between the cartographic model and the digital model. After that, Monte-Carlo simulation and interval arithmetic are used do propagate DEM uncertainty to the derived Digital Slope Model (DSM), based on Horn's method. Finally, slope classes are derived from DSM: I (0%-5%), II (5%-15%), III (15%-30%), IV (30%-40%) and V (>40%). Slopes classes uncertainty is also evaluated.

This paper is organized in the following way: in section 2 the problem of DEMs uncertainty modeling is studied and the main uncertainty propagation methods are presented; in section 3 will be presented the methodology used to implement and compare the uncertainty propagation methods; in section 4 the numerical results are showed; finally, in section 5, the conclusions will be presented and some future developments will be suggested.

MODELS FOR DEM UNCERTAINTY AND THEIR PROPAGATION

An important problem in uncertainty study is to set a model which expresses the spatial and statistic distribution of error close to reality. The gaussian model is often used in this context, where the true value is estimated by the sample mean and the standard deviation expresses a measure of uncertainty.

In this paper, we will suppose that DEM uncertainty can me modeled adding to every grid elevation a disturbing term from a random field spatially independent from neighbors. Let $z(x,y)$ be a DEM elevation in (x,y) position, we get:

$$z(x,y) = \tilde{z}(x,y) + N(\mu_z, \sigma_z), \tag{1}$$

where $\tilde{z}(x,y)$ is the DEM elevation generated by interpolation from the cartographic model and $N(\mu_z, \sigma_z)$ represents the gaussian independent random variable with zero mean ($\mu_z = 0$) and standard deviation σ_z .

Uncertainty in the cartographic model and in the DEM

Terrain cartographic models are usually represented by elevation contours and elevation points (spot heights). They are a common source of topographic data used to generate DEMs. When contours and elevation points are manually generated by photogrammetric stereoplotting, it is usually accepted that the elevation uncertainty of cartographic model is expressed by the following criteria: 90% of elevation of control points should not have a mean square error bigger than half of the contour interval (see [18]). Therefore, if Δz is the contour interval and assuming that elevation error in the cartographic model is gaussian with zero mean, we get:

$$\sigma_z = \frac{1}{2 \times 1.645} \Delta z. \tag{2}$$

To ensure that the elevation represented in the cartographic model keeps the same in DEM it is necessary to have equivalence between both models. Two models are equivalent if they contain “potentially” the same information ([7]). That equivalence can be expressed in terms of explicit information in both models (the coordinates of a point in those models, for example) or implicitly (terrain morphology or slope, for example). Only in these conditions we can assume that DEM uncertainty is similar to the cartographic model uncertainty.

The generation of a DEM from a cartographic model should be done using a specialized interpolator. That interpolator should ensure cartographic equivalence between the two models ([7] e [8]). A method to evaluate the cartographic properties of an interpolator is explained in [9]. In this study it was applied the modified elastic grid method, described in [8], to generate the DEM. When a general interpolator is used, the equivalence of models is not assured. Therefore, the uncertainty of DEM generation should also be included.

The mathematic model of terrain surface is usually a function $z = f(x,y)$, of class C^1 and with domain $D \subset \mathbf{R}^2$. To generate the DSM, the slope in every position (x,y) is given by the expression

$$S = \sqrt{f_x^2 + f_y^2}, \tag{3}$$

where f_x and f_y are the partial derivatives of f. The slope is usually derived from a digital terrain model defined over a grid (the DEM). In this context, the partial derivatives are estimated using finite differences. Accordingly, a 3x3 window of elevations centered in (x,y) is used (figure 1). This window will move over the entire grid, generating the DSM.

A common method used to estimate the slope is the Horn formula, where partial derivatives of central cell are given by

$$f_x = \frac{(z_1 + 2z_4 + z_7) - (z_3 + 2z_6 + z_9)}{8\Delta x} \tag{4}$$

and

$$f_y = \frac{(z_1 + 2z_2 + z_3) - (z_7 + 2z_8 + z_9)}{8\Delta y}, \tag{5}$$

where Δx and Δy represent the grid cell size (see [1] and [5]).

z_1	z_2	z_3
z_4	z_5	z_6
z_7	z_8	z_9

Figure 1: Numeration scheme of elevations in a 3x3 window of DEM.

The slope is a geomorphologic parameter used in many different situations of spatial modeling. Consequently, their classification is not fixed, depending on the application. Therefore, the number of slope classes and the interval limits can be quite different. To illustrate our study, we will use the following slope classification:

class	I	II	III	IV	V
Slope (%)	0-5	5-15	15-30	30-40	>40

Table 1: Slope classes.

Methods to model uncertainty propagation from DEM to DSM

The determination of DEM elevations is a procedure exposed to several sources of uncertainty, from data acquisition to model generation. There are several methods which can be used to evaluate the effect of the propagated uncertainty from elevation to slope. Each of those methods is based in a different uncertainty model.

Variance propagation method: Most of the probabilistic uncertainty models are based in the gaussian distribution. Setting the mathematical model

$$y = f(x_1, x_2, ..., x_n),$$

the variance propagation formula is

$$\sigma_y^2 \approx f_1^2 {\sigma_1}^2 + f_2^2 {\sigma_2}^2 + \ldots + f_n^2 {\sigma_n}^2, \tag{6}$$

where $f_1, f_2, \ldots, f_n$ are the partial derivatives of f in order to $x_1, x_2, \ldots, x_n$, respectively.

In the particular case of slope, we have first to find the variance of derivative approximations given by (7) and (8) from elevations variance. We will set the same variance for all elevations: $\sigma_z(x, y) = \sigma_z, \forall (x, y) \in D$. Consequently, we get

$$\sigma_{f_x}^2 = \frac{3}{16\Delta x} \sigma_z^2 \text{ and } \sigma_{f_y}^2 = \frac{3}{16\Delta y} \sigma_z^2. \tag{9}$$

From the derivatives variance, we find easily the slope variance, which is

$$\sigma_s^2 = \frac{{f_x}^2}{{f_x}^2 + {f_y}^2} \sigma_{f_x}^2 + \frac{{f_y}^2}{{f_x}^2 + {f_y}^2} \sigma_{f_y}^2. \tag{10}$$

If $\Delta = \Delta x = \Delta y$ is the cell size of DEM, we have

$$\sigma_S = \frac{\sqrt{3}}{4\Delta} \sigma_z \tag{11}$$

Monte-Carlo Method: Monte-Carlo method is based on simulated realizations of a random process defined by certain parameters, previously estimated. In this particular problem, we will start by generate a certain number of realizations for the DEM constrained to prescribed uncertainty parameters. It will be generated a DSM from every DEM realization. The set of DSM realizations will be statistically analyzed.

Interval Arithmetic: The interval arithmetic approach has been used with success in uncertainty modeling for some decades ([16]). As the designation suggests, in interval arithmetic real numbers are replaced by real intervals to express uncertain quantities. The processing of these intervals is done using operators whose arguments are intervals ([15] e [17]).

Let $[z_i] \square [\underline{z}_i, \overline{z}_i] \subset \mathbf{R},\ i = 1, 2, \ldots$ be closed real intervals. Elementary interval operations are defined by:

$$\begin{aligned} &[z_i]+[z_j]=[\underline{z}_i+\underline{z}_j,\overline{z}_i+\overline{z}_j],\\ &[z_i]-[z_j]=[\underline{z}_i-\overline{z}_j,\overline{z}_i+\underline{z}_j],\\ &\lambda[z_i]=\begin{cases}[\lambda\underline{z}_i,\lambda\overline{z}_i], & \lambda\ge 0\\ [\lambda\overline{z}_i,\lambda\underline{z}_i], & \lambda<0.\end{cases}\end{aligned} \tag{12}$$

The underlying principle is that an operation result is the smaller interval which includes all the possible outcomes for all the values in interval inputs.

We will go to apply this model to slope uncertainty, suggested by [18]. Therefore, uncertainty in elevations will be expressed by specifying the elevation z_i in every DEM cell by an interval $[z_i] = [\underline{z}_i, \overline{z}_i]$. So, given an elevation z_i from a cell, an interval $[z_i]$ is specified such that it includes the true elevation of topographic surface with certain degree of confidence.

Consequently, for partial derivatives (4) e (5), using elementary operations (12), we get:

$$\begin{aligned} [f_x] &= \frac{([z_1]+2[z_4]+[z_7])-([z_3]+2[z_6]+[z_9])}{8\Delta x}\\ [f_y] &= \frac{([z_1]+2[z_2]+[z_3])-([z_7]+2[z_8]+[z_9])}{8\Delta y}\end{aligned} \tag{13}$$

After evaluate partial derivatives of elevation in interval form, the terrain slope for a cell is given in interval form by

$$[S] = \sqrt{\left([f_x]\right)^2 + \left([f_y]\right)^2}\,. \tag{14}$$

Therefore, we need to calculate the square and the square root of an interval. The square of an interval $[z_i] = [\underline{z}_i, \overline{z}_i]$ is given by

$$\left([z_i]\right)^2 = \begin{cases}[\underline{z}_i^{\,2}, \overline{z}_i^{\,2}], & \underline{z}_i \ge 0\\ [\overline{z}_i^{\,2}, \underline{z}_i^{\,2}], & \overline{z}_i < 0\\ [0, \max\{\underline{z}_i^{\,2}, \overline{z}_i^{\,2}\}], & \text{otherwise.}\end{cases} \tag{15}$$

The square root of an interval is simply

$$\sqrt{[z_i]} = [\sqrt{\underline{z}_i}, \sqrt{\overline{z}_i}], \quad \underline{z}_i \ge 0\,. \tag{16}$$

METHODOLOGY

Characterization of test area

The study area was extracted from a $1:10000$ scale digital topographic map. The contour interval is 5 meters and the study area is about 20 square kilometers. The topography of this area can be characterized numerically by the standard deviation of elevations and of slope [6]. The histogram of elevations is given in Figure 2-a). Figure 2-c) shows the shaded relief of the terrain. The histogram of DEM slope is shown in Figure 2-b). Figure 2-c) shows slope classes.

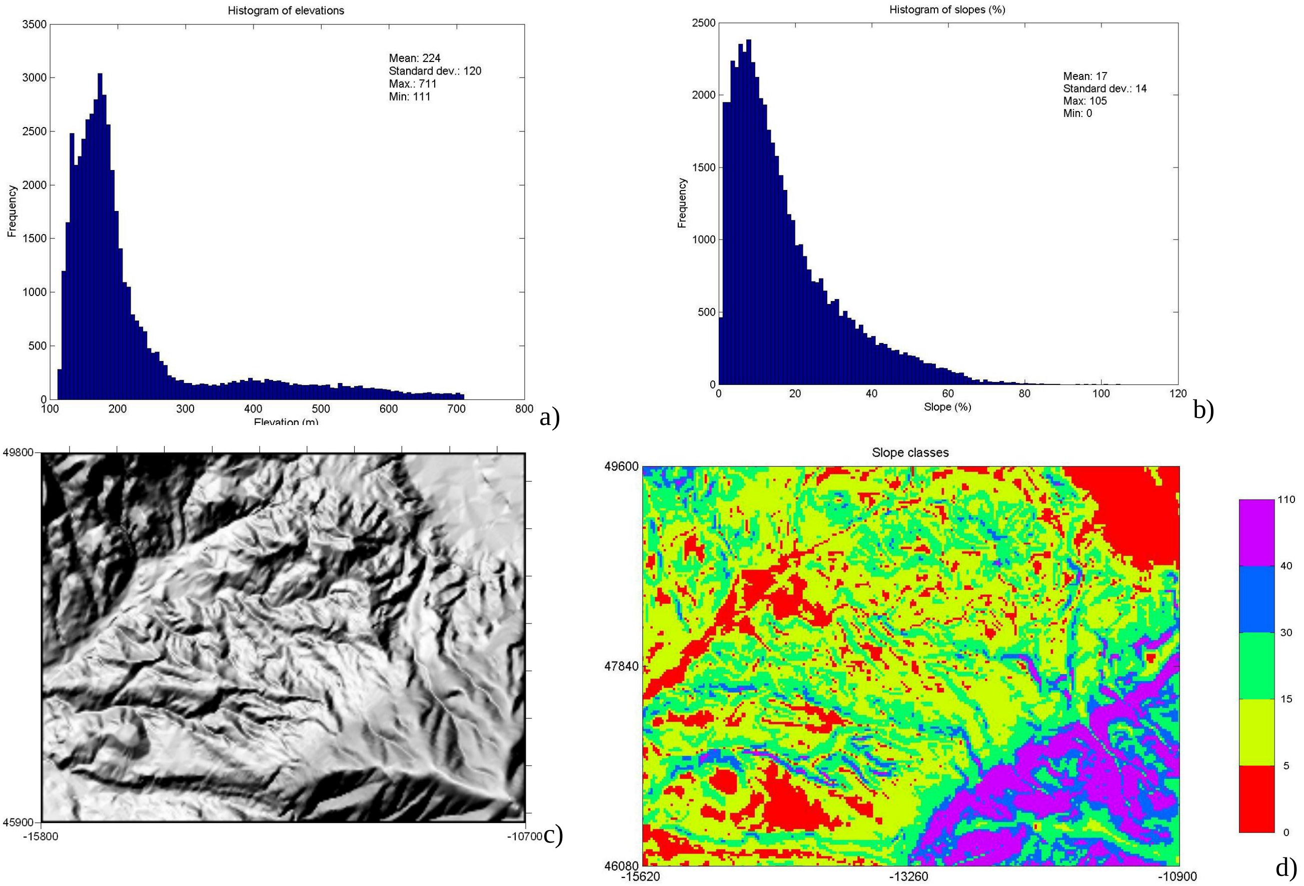

Figure 2: Characteristics of study area: a) Histogram of elevations; b) Histogram of slopes; c) Shaded DEM; d) Slope classes.

Uncertainty propagation and comparison

Setting expression (1) as the model for elevation uncertainty, where standard deviation in given by (2), we will analyze the effect of that uncertainty in slope. Starting by combining relations (2) and (11), we get

$$\sigma_S = 0.132 \frac{\Delta z}{\Delta}, \tag{17}$$

showing that slope uncertainty increases for finer resolution of the cartographic model or for coarser resolution of DEM. Particularly, if we set the same resolution for both models, DSM uncertainty will be around 13%. Standard deviation σ_S decreases for coarser DEM resolutions but that implies less detail in DEM in relation to the cartographic model.

However, later analysis is quite simplistic and can not express spatial variation of uncertainty. To overcome that limitation we will apply the two other methods presented above. To apply Monte-Carlo method to our study area, one hundred DEM realizations were generated using model (1) and a DSM was derived from each realization. To use Interval Arithmetic we set a 90% confidence interval for elevations in order to keep the confidence level prescribed for the cartographic model. Interval computations were applied over elevation intervals to derive slope intervals, using *Intlab*™ toolbox running on *Matlab*™.

Slope classification was executed after DSM generation following Table 1. To assign one slope value to each cell, the mean value of all slope realizations was used in Monte-Carlo method and the interval midpoint was used in the case of Interval Arithmetic.

The comparison of DEM uncertainty propagation methods in slope classification was done by two different approaches: visually, using both slope classes' maps, and analytically, using confusion matrix.

RESULTS

The visual comparison of results from both methods of elevation uncertainty propagation to slope classification is illustrated in Figure 3. Figure 3-a) shows uncertainty propagation modeled by Interval Arithmetic and Figure 3-b) shows the same for Monte-Carlo method. Looking carefully at the figures it can be noticed that the results for both methods are quite close. The exception is class I (0-5%) represented in red in both figures, where in the case of Interval Arithmetic seems to occupy a larger area.

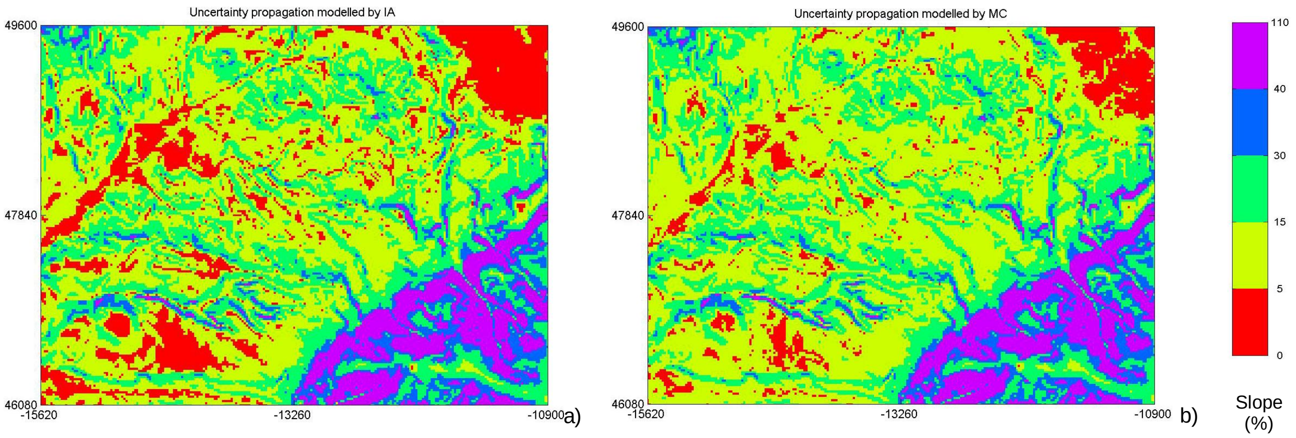

Figure 3: Comparison of uncertainty propagation methods: a) uncertainty propagation using Interval Arithmetic (IA); b) uncertainty propagation using Monte-Carlo (MC) method.

Visual results are confirmed by the confusion matrix in Table 2. The amount of pixels classified in the same way by both methods is in fact 92%. Furthermore, if class I is ignored, the amount of pixels classified in the same way for each class is very high (more than 97%). This means that both methods of uncertainty propagation produce almost the same results for classes II, III, IV e V.

Slope classes with uncertainty – Interval Arithmetic	Slope classes with uncertainty - Monte-Carlo method							
	Classes	I	II	III	IV	V	Total	PCC
	I	3755	3234	0	0	0	6989	53.73
	II	0	18266	496	0	0	18762	97.36
	III	0	0	12654	89	0	12743	99.3
	IV	0	0	0	3770	43	3813	98.87
	V	0	0	0	0	3882	3882	100
	Total	3755	21500	13150	3859	3925	46189	91.64

Table 2: Confusion matrix comparing uncertainty in slope classification using Monte-Carlo method and Interval Arithmetic.

However, for class I we have only 54% of agreement. Looking at the slope histogram in Figure 2-b, it can be noticed that there is a strong variation around the value 5, which induces instability when class limit is close to that region of histogram. For example, if we change classes I and II to 0-7% and 7-10%, respectively, we will get the values 82% and 96% of agreement, respectively.

CONCLUSIONS

In this paper we showed that Interval Arithmetic is a valid alternative to evaluate elevation uncertainty propagation to slope classification. However, all presented propagation methods have qualities and limitations.

The variance propagation method needs partial derivation of slope as a mathematical function depending on elevation. Since that function is not complex, it is not a difficult task, and it is done only once (symbolic differentiation software can be used in that task, like *Mathematica*™, *Maple*™ or *Matlab*™). The main characteristic of this propagation method is their simplicity. It can be a quality because it is easy to apply and a limitation since it implies too much simplification in the uncertainty model. For example, DEM elevation errors as to be gaussian with zero mean. Furthermore, it can not model spatial uncertainty distribution.

The main qualities of Monte-Carlo method are the flexibility of application and the statistical basis which is widely known and accepted in many fields of science. The limitation is the need of intensive computational work which turns it hard to implement in spatial analyses with GIS.

One advantage of Interval Arithmetic is the simplicity which allows an easy understanding about DEM uncertainty propagation to slope classification. Besides, the implementation of Interval Arithmetic in GIS is not too complex and does not need so much computation power as Monte-Carlo method. The main limitation is the overestimation of uncertainty caused by the lack of distribution of confidence, giving the same chance of occurrence to every value inside the interval.

Further studies about the application of Interval Arithmetic in GIS uncertainty propagation will be carried out in future. For example, to build a more complex elevation uncertainty model using GPS control points and geostatistical analysis.

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