CONFORMAL PROJECTION WITH MINIMAL DISTORTIONS

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Abstract
The paper presents results of research on Chebyshev projection. This projection has the least distortions among other conformal projections. The main aim of research was the elaboration of algorithms of construction such projections and construction of Chebyshev projections of selected areas. There is also, in the paper, comparison between Chebyshev projection and other conformal projection applied in Poland.
The paper consists of two main parts; theoretical part, which contains: basic formulas, some problems related to application numerical method in Chebyshev projection, algorithms and practical part, which contains: maps presenting distortions, comparison Chebyshev projection with other conformal map projection of Poland.

1. INTRODUCTION

One of the main aim of cartographers and mathematicians who dealt with map projections was to obtain projection with minimal distortion. We can find many examples of criterions which allow to get minimum distortion in projected area. It is worth to mention such criterions as: Jordan’s, Tissot’s, Airy’s and Chebyshev’s. The last one is most interesting because it is proofed that conformal map projection satisfied that criterion has the least distortions among other conformal projections of a given area.

P.L. Chebyshev (1821-1894) as a measure of distortion used natural logarithm of an elementary scale in the following

\[ z = \ln m \] (1)

Based on this measure, Chebyshev gave his own criterion as follows

\[ \ln \left( \max_{(u,v) \in F} (\text{max } m) \right) - \ln \left( \min_{(u,v) \in F} (\text{min } m) \right) = \min_{A} \] (2)

where \( F \) is a given area, \( u,v \) are parameters and \( A \) - azimuth at the point \((u,v)\). It means that we get the best projection when the difference between maximum and minimum of the logarithm of a scale reach minimum in the given area \( F \).

In 1853 Chebyshev formulated theorem about “the best” map projection. This theorem states that a map projection of a given area has the minimum difference between maximum and minimum distortions when the elementary scale along boundary of a given area is constant.

In 1896 other Russian scientist D. A. Grawe proved the Chebyshev’s theorem for conformal projections. According to that proof if the elementary scale along boundary of a given area is constant the difference between maximum and minimum natural logarithm of the scale in a given area is minimum.

In spite of evident advantages Chebyshev projection has not been wide-spread, probably in regard to numeric problems appearing in such projections. Especially, some difficult task seemed to be description of boundaries of a projected area.
In literature we can find a few examples of projections satisfying Chebyshev criterion. However they refer to simplified boundary structures, mainly geodetic circles or ovals.

H. Urmayev (Urmayev 1969) constructed projection of a hemisphere limited by meridians $\lambda=-\pi/2$ and $\lambda=+\pi/2$. The value of the elementary scale reaches maximum $m=1$ on the boundary and minimum $m=1/2$ in the center of the area.

In 1953, O. M. Miller prepared modified stereographic projection for Europe and Africa. His resulting maps have oval lines of constant scale. This projection is called the Miller Oblated Stereographic (Snyder 1987).

Lee applied the Chebyshev criterion to the oblique Stereographic projection for map of the Pacific Ocean (Snyder 1987).

Snyder (Snyder 1987) prepared modified-stereographic Conformal projection of 48 United states, bounded by near-rectangle of constant scale.

B. Gdowski (Gdowski 1969) elaborated the analytical method of construction Chebyshev projection of geodetic circles. He solved the Dirichlet’s problem by means of the orthonormal system of harmonic function. In other work B. Gdowski (Gdowski 1971) described simplified method of construction of Chebyshev’s projection of geodetic circle using power series limited to the third order.

Finally L. Bugayevski (Bugayevski 1986) gave the scheme of construction Chebyshev projection by means of numerical methods.

In the paper there are presented method construction and examples of Chebyshev projection of limited areas of an ellipsoid. We can also find some problems concerning Chebyshev projection in the following works: (Balcerzak, Pędzich 1999), (Pędzich 1999), (Pędzich 2002).

2. THE BASES OF CONSTRUCTION THE CHEBYSHEV PROJECTION

Conformal map projections, also the Chebyshev projection, bases on analytic functions of complex variable. Such a function, in conformal projection, represents dependence between isometric parameters of original surface and isometric parameters of image surface. We can obtain this function by solving proper differential equations. In order to solve this equations we have to find function describing elementary scale and convergence.

2.1. Conformal map projection – projecting functions, elementary scale and convergence of meridians

A conformal projection may be presented in general form by means of complex function

$$x + iy = f(q + il),$$ (3)

where $q$, $l=L-L_0$ are isometric parameters on an ellipsoid and $x,y$ are isometric parameters on image surface. The function (3) is an analytic function satisfying Cauchy-Riemann conditions.

Function (3) may be expand in the following power series

$$x + iy = \sum_{k=1}^{\infty} (a_k + ib_k)(\xi + i\eta)^k,$$ (4)

in which $a_k$ and $b_k$ are real numbers, while variables $\xi$ and $\eta$ combine with arguments $q$ and $l$ according to the following relationship

$$\xi = q - q_0 \quad \eta = l, \quad q_0 = \text{const.}$$ (5)

For the power series (4) we can apply the following recurrent formula
If we separate series (4) on real and imaginary part, we will obtain the following

\[
\begin{align*}
\left(\xi + i\eta\right)^k &= \Psi_k + i\theta_k \\
\Rightarrow \quad \Psi_k &= \xi\Psi_{k-1} - \eta\theta_{k-1} \\
\theta_k &= \xi\theta_{k-1} + \eta\Psi_{k-1}
\end{align*}
\]

(6)

If the projection (3) represented by series of the form of (7) is supposed to be also a symmetrical one according to axial meridian \(L=L_0\), the following dependences must come about

\[
x(q, l) = x(q, -l), \quad y(q, l) = -y(q, l)
\]

(8)

Harmonic polynomials \(\Theta_k, k = 1, 2, 3, \ldots\), in (7) depend merely on odd power \(\eta\). Thus condition of symmetry (8) in (7) leads to reduction \(b_k = 0, k = 1, 2, 3, \ldots\), and therefore we obtain the following

\[
x = \sum_{k=1}^{\infty} a_k \Psi_k, \quad y = \sum_{k=1}^{\infty} a_k \theta_k
\]

(9)

The elementary scale in projection (3) is determined by the formula

\[
\left[ m = \frac{\mu}{r} \right] \Rightarrow (\ln m = \ln \mu - \ln r)
\]

(10)

where \(\mu = |f|, \quad r = N\cos B\).

Since \(\ln \mu\) is a harmonic function of variables \(q\) and \(l\) we may look for it in the form of the following series

\[
\ln \mu = \sum_{k=0}^{\infty} \hat{a}_k \Psi_k + \sum_{k=1}^{\infty} \hat{b}_k \theta_k
\]

(11)

It is a sum of linear combinations of harmonic polynomials

\[
\begin{align*}
\Psi_k &= \xi\Psi_{k-1} - \eta\theta_{k-1} \\
\theta_k &= \xi\theta_{k-1} + \eta\Psi_{k-1}
\end{align*}
\]

(12)

In conformal map projections the following partial derivatives occur
\[
\begin{align*}
\begin{cases}
x_q = \mu \cos \gamma \\
y_q = \mu \sin \gamma
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
x_L = -\mu \sin \gamma \\
y_L = \mu \cos \gamma
\end{cases}
\end{align*}
\]
Based on the integration condition of system equation (13), (14)
\[
\begin{align*}
\left\{x_q\right\}_L &= (x_q)_L, \quad \left\{y_q\right\}_L = (y_q)_L
\end{align*}
\]
we can write
\[
\begin{align*}
\gamma_q &= -(\ln \mu)_L, \quad \gamma_L = (\ln \mu)_q
\end{align*}
\]
Function \(\gamma\) and \(\ln \mu\) are harmonic ones. Based on (11) and (16) we obtain the following integrals
\[
\begin{align*}
\gamma &= \int (\ln \mu)_L dq + C_1(L) = \int \left[ \sum_{k=1}^{\infty} \hat{a}_k (\theta_k)_q - \sum_{k=1}^{\infty} \hat{b}_k (\Psi_k)_q \right] dq + C_1(L) \\
\gamma &= \int (\ln \mu)_q dL + C_2(q) = \int \left[ \sum_{k=0}^{\infty} \hat{a}_k (\theta_k)_L - \sum_{k=1}^{\infty} \hat{b}_k (\Psi_k)_L \right] dL + C_2(q)
\end{align*}
\]
If we assume in (17) that \(C_1(L) = C_2(q) = 0\) the following comes about
\[
\begin{align*}
\gamma &= \sum_{k=1}^{\infty} \hat{a}_k \theta_k - \sum_{k=1}^{\infty} \hat{b}_k \Psi_k
\end{align*}
\]
therefore convergence angle of meridians \(\gamma\) in conformal projection is expressed by the same numerical coefficients which occur in (11).
If we assume that
\[
\begin{align*}
x &= \sum_{k=1}^{\infty} a_k \Psi_k - \sum_{k=1}^{\infty} b_k \theta_k \\
y &= \sum_{k=1}^{\infty} a_k \theta_k + \sum_{k=1}^{\infty} b_k \Psi_k
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
x_q = \sum_{k=1}^{\infty} a_k (\psi_k)_q - \sum_{k=1}^{\infty} b_k (\theta_k)_q \\
y_q = \sum_{k=1}^{\infty} a_k (\theta_k)_q + \sum_{k=1}^{\infty} b_k (\psi_k)_q
\end{cases}
\end{align*}
\]
and if we take into account in (19) that
\[
\left(\psi_k\right)_q = (\theta_k)_L = k \Psi_{k-1}, \quad (\theta_k)_q = -(\psi_k)_L = k \theta_{k-1}
\]
on the basis of (13), we will obtain the following system

\[ \sum_{k=1}^{\infty} k a_k \Psi_k = \mu \cos \gamma \]
\[ \sum_{k=1}^{\infty} k b_k \theta_k = \mu \sin \gamma \]

That system enables to calculate coefficients in any conformal projection.

3. THE ALGORITHM OF CONSTRUCTION OF CHEBYSHEV PROJECTION BY MEANS OF POWER SERIES

1. Description of boundary of a given area.
   Some problem is how to describe boundary of a given area. We can inscribe a polygon into boundary or we can simplify the boundary of a given area to the shape of a circle or an ellipse. Then we can calculate coordinates for each point situated on circle or ellipsoid using any method of coordinate transfer along geodetic line of ellipsoid. We have to solve direct aspect of coordinate transfer, having coordinates of central point, length of geodetic line and the azimuth of geodetic line at central point.

2. Calculation of geodetic coordinates \((B,L)\) of graticule nodes.

3. Calculation of isometric coordinates \((q,l)\) of boundary points of a given area and nodes of graticule on an ellipsoid.

4. Calculation of harmonic polynomials (12) for boundary points.

3. Calculation of coefficients in series (11) and (18) approximating elementary scale and convergence.
   For each point \((q_i,l_i)\), \(i=1,2,3,...\) situated on the boundary of given area the following equation will be formed

\[ \sum_{k=0}^{\infty} \hat{a}_k \Psi_k + \sum_{k=1}^{\infty} \hat{b}_k \theta_k - \ln r_i = 0 \]

implied by (10) and (11).
   To simplify the problem we may assume that scale \(m=1\) on the boundary.
   Solution of system equation (22) by the least square method provides numerical values of coefficients \(\hat{a}_k, \hat{b}_k\), \(k = 1, 2, 3, ...\)
   Thus it is possible to determine function \(\mu\) and convergence \(\gamma\) at any point of a given area.

4. Calculation of convergence and elementary scale at graticule nodes.
   The elementary scale \(m = \frac{\mu}{r}\) at any point may be calculated based on following formula

\[ \mu = e^{\ln \mu} = e^{\sum_{k=1}^{\infty} \hat{b}_k \theta_k + \sum_{k=0}^{\infty} \hat{a}_k \Psi_k} \]

and the convergence \(\gamma\) using (18).

5. Calculation of coefficients in series approximating projecting functions.
   For each point of coordinates \(q,l\) whose parameters \(\mu\) and \(\gamma\) are known, we set up equation (21) with unknowns \(a_k, b_k\), \(k = 1, 2, 3, ...\)
   After solving of system (21) we obtain the numerical values of coefficients occurring in power series (7).

4. CHEBYSHEV PROJECTION OF POLAND

For area of Poland two case of Chebyshev projection has been prepared, asymmetric one and symmetric one. Figure 1 presents graphic illustration of distortions in asymmetric projection of Poland.
   In asymmetric case the boundary of Poland was simplified to a polygon.
Figure 2 presents graphic illustration of distortions in symmetric case of Chebyshev projection of Poland. In symmetric case of projection the procedure of determining boundary points was simplified. Boundary was approximated by geodetic circle. To calculate coordinates of boundary points, method of transfer of ellipsoidal coordinates in direct problem was used.

The maximum distortion in asymmetric projection is about 35 cm/km and in symmetric one is equal to 50 cm/km.
We can compare Chebyshev projection with other conformal projections of Poland. Figure 3 presents Roussilhe projection of Poland. The maximum distortion in that projection is about 50 cm/km. Roussilhe projection is very similar to symetric Chebyshev projection of Poland (Fig. 2).

Figure 4 presents Gauss-Krüger projection of Poland. There is one zone for entire area. In this projection we have maximum distortion equal to 70 cm/km. It is much more than in Chebyshev Projection.
5. SOME EXAMPLES OF CHEBYSHEV PROJECTIONS WHEN A BOUNDARY OF A PROJECTED AREA IS SIMPLIFIED TO THE SHAPE OF AN CIRCLE OR AN ELLIPSE

Figure 5 presents graphic illustration of distortions in symmetric projection of Spain. The boundary was approximated by geodetic circle.

If a given area is stretched along meridian or parallel, then an ellipse may be used to approximate boundary. Figure 6 presents graphic illustration of distortions in symmetric projection of Portugal. The ellipse is oriented along meridian $L=-8^\circ$. 
Figure 7 presents graphic illustration of distortions in symmetric projection of Turkey. The ellipse is oriented along parallel $B=39^\circ$.

Fig. 7 Chebyshev projection of Turkey

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Pawel Pedzich was born at Wegrow, Poland on August 2, 1971. He graduated Warsaw University of Technology, Faculty of Geodesy and Cartography, specialization cartography, in 1997. In this year he started doctoral studies at Warsaw University of Technology. In 1998 he did academic practices at Naval Academy in Gdynia. In 2002 he obtained his doctor's degree. The thesis was on map projections. In 2003 he started to work, as a lecturer, at Institute of Photogrammetry and Cartography, Warsaw University of Technology. In the same year he also started to work as a lecturer at Institute of Civil Engineering and Geodesy, Military University of Technology in Warsaw. He lectures mathematical cartography at Warsaw University of Technology, cartography, topography, geographic information systems at Military University of Technology. He is an author of several publications on map projections, presentations on national conferences. He leaded several research projects.